Bipartite graphs with the maximum sum of squares of degrees*

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Abstract

In this paper we determine all the bipartite graphs with the maximum sum of squares of degrees among the ones with a given number of vertices and edges.

Keywords: Bipartite graphs; Sum of squares of degrees; Extremal graphs

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1 Introduction

All graphs considered here are finite, undirected and simple. For terminology and notation not defined here we follow those in Bondy and Murty [3].

In this paper we study an extremal problem on bipartite graphs: among all bipartite graphs with a given number of vertices and edges, find the ones where the sum of squares of degrees is maximum.

The corresponding problem for general graphs has been studied in [1, 2, 7]. For all graphs with a given number vertices and edges, Ahlswede and Kanota [1] first determined the maximum sum of squares of degrees. Boesch et al. [2] proved that if the sum of squares of degrees attains the maximum, then the graph must be a threshold graph (See the definition in [6]). They constructed two threshold graphs and proved that at least one of them is such an extremal graph. Peled et al. [7] further studied this problem and showed that, if a graph has the maximum sum of squares of degrees, then it must belong to one of the six particular classes of threshold graphs.

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For the family of bipartite graphs with a given number of vertices and edges and the size of one partite side, Ahlswede and Kanota [1] determined a bipartite graph such that the sum of squares of its degrees is maximum. Recently, Cheng et al. [4] determined the maximum sum of squares of degrees for bipartite graphs with a given number of vertices and edges.

While the problem of finding all the graphs with a given number of vertices and edges where the sum of squares of degrees is maximum is still unsolved, we give a complete solution to the problem of finding all the bipartite graphs with a given number of vertices and edges where the sum of squares of degrees is maximum in this paper. In Section 2 we present some notation and lemmas that will be used later and in Section 3 give the main results and the proof.

2 Notation and lemmas

Let x be a real number. We use $\lfloor x \rfloor$ to represent the largest integer not greater than x and $\lceil x \rceil$ to represent the smallest integer not less than x. The sign of x, denoted by $sgn(x)$, is defined as 1, -1, and 0 when x is positive, negative and zero, respectively.

Let n, m and k be three positive integers. We use $B(n,m)$ to denote a bipartite graph with n vertices and m edges, and $B(n,m,k)$ to denote a $B(n,m)$ with a bipartition (X,Y) such that $|X| = k$. By $\mathscr{B}(n,m)$ we denote the set of graphs of the form $B(n,m)$ and $\mathscr{B}(n,m,k)$ the set of graphs of the form $B(n,m,k)$.

Suppose that n, m and k are three integers with $n \geq 2$, $0 \leq m \leq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ and $1 \leq k \leq n-1$. Let $m = qk + r$, where $0 \leq r < k$. Then $B^l(n,m,k)$ is defined as a bipartite graph in $\mathscr{B}(n,m,k)$ such that q vertices in Y are adjacent to all the vertices of X and one more vertex in Y is adjacent to r vertices in X if $r > 0$.

We use $\mathscr{G}(n,m)$ to denote the family of graphs with n vertices and m edges. Given an integer $t \geq 2$, and a graph $G \in \mathscr{G}(n,m)$, let

$$\sigma_t(G) = \sum_{v \in V(G)} (d(v))^t.$$

The following result is due to Ahlswede and Kanota [1].

Lemma 1 (Ahlswede and Kanota [1]). *Let n, m and k be three integers with $n \geq 2$, $0 \leq m \leq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ and $\lceil \frac{n}{2} \rceil \leq k \leq n-1$. Suppose that $m = qk + r$, where $0 \leq r < k$. Then $\sigma_2(B^l(n,m,k))$ attains the maximum value among all the graphs in $\mathscr{B}(n,m,k)$.*

With this result, Cheng et al. [4] obtained the following

Lemma 2 (Cheng, Guo, Zhang and Du [4]). *Let n, m be two integers with $n \geq 2$, $n \leq m \leq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ and $k_0 = max\{k | m = qk + r, 0 \leq r < k, \lceil \frac{n}{2} \rceil \leq k \leq n - q - sgn(r)\}$. Then $\sigma_2(B^l(n, m, k_0))$ attains the maximum value among all the bipartite graphs in $\mathscr{B}(n, m)$.*

For general graphs with few edges, Ismailescu and Stefanica [5] got the following result.

Lemma 3 (Ismailescu and Stefanica [5]). *Let n, m and t be three integers with $n \geq 2$, $m \leq n-2$ and $t \geq 2$. Suppose that $\sigma_t(G^*)$ attains the maximum value among all the graphs in $\mathscr{G}(n, m)$. Then $G^* \cong K_{1,m} \cup S_{n-m-1}$, the star with m edges plus $n - m - 1$ isolated vertices, except the case $t = 2$ and $m = 3$, where both $\sigma_t(K_{1,3} \cup S_{n-4})$ and $\sigma_t(K_3 \cup S_{n-3})$ attains the maximum.*

Let B be a bipartite graph. We use $\overline{B}$ to denote the bipartite graph on the same partition as B such that two vertices in $\overline{B}$ are adjacent if and only if they are not adjacent in B.

Lemma 4. *Let B be a bipartite graph in $\mathscr{B}(n, m, k)$. Then $\sigma_2(B)$ attains the maximum value among all the graphs in $\mathscr{B}(n, m, k)$ if and only if $\sigma_2(\overline{B})$ attains the maximum value among all the graphs in $\mathscr{B}(n, k(n-k) - m, k)$.*

Proof. Let (X, Y) be the bipartition of B. Suppose that $X = \{x_1, x_2, \ldots, x_k\}$ and $Y = \{y_1, y_2, \ldots, y_{n-k}\}$. Denote the degree of x_i in $\overline{B}$ by $\overline{d}(x_i)$ for $i = 1, 2, \ldots, k$ and the degree of y_j in $\overline{B}$ by $\overline{d}(y_j)$ for $j = 1, 2, \ldots, n-k$. Then we have

$$d(x_i) + \overline{d}(x_i) = n - k \ \text{ for } i = 1, 2, \ldots, k, d(y_j) + \overline{d}(y_j) = k \ \text{ for } j = 1, 2, \ldots, n-k,$$

and

$$\sum_{i=1}^{k} \overline{d}(x_i) = \sum_{j=1}^{n-k} \overline{d}(y_j) = k(n-k) - m.$$

Therefore,

$$\begin{aligned}
\sigma_2(B) &= \sum_{i=1}^{k} d(x_i)^2 + \sum_{j=1}^{n-k} d(y_j)^2 \\
&= \sum_{i=1}^{k} (n - k - \overline{d}(x_i))^2 + \sum_{j=1}^{n-k} (k - \overline{d}(y_j))^2 \\
&= k(n-k)^2 - 2(n-k) \sum_{i=1}^{k} \overline{d}(x_i) + \sum_{i=1}^{k} \overline{d}(x_i)^2 \\
&\quad + (n-k)k^2 - 2k \sum_{j=1}^{n-k} \overline{d}(y_j) + \sum_{j=1}^{n-k} \overline{d}(y_j)^2 \\
&= n(2m + k^2 - nk) + \sum_{i=1}^{k} \overline{d}(x_i)^2 + \sum_{j=1}^{n-k} \overline{d}(y_j)^2 \\
&= n(2m + k^2 - nk) + \sigma_2(\overline{B}).
\end{aligned}$$

The result follows immediately. □

3 Main results

We first determine the bipartite graphs with few edges where the sum of squares of degrees is maximum.

Theorem 1. *Let n, m be two integers with $n \geq 2$ and $0 \leq m \leq n-1$. Suppose that $\sigma_2(B^*)$ attains the maximum value among all the graphs in $\mathscr{B}(n, m)$. Then $B^* \cong K_{1,m} \cup S_{n-m-1}$.*

Proof. From Lemma 2 we know that $\sigma_2(B^l(n, m, k_0))$ attains the maximum value among all the bipartite graphs in $\mathscr{B}(n, m)$, where $k_0 = max\{k|m = qk + r, 0 \leq r < k, \lceil \frac{n}{2} \rceil \leq k \leq n - q - sgn(r)\}$. So we have $\sigma_2(B^l(n, m, k_0)) = \sigma_2(B^*)$. We distinguish two cases.

Case 1. $0 \leq m \leq n-2$.

Let $m = q_0 k_0 + r_0$, where $0 \leq r_0 < k_0$. Then we can conclude $k_0 = n-1$, $q_0 = 0$ and $r_0 = m$. Hence, $B^l(n, m, k_0) = K_{1,m} \cup S_{n-m-1}$. By Lemma 3 we know that $K_{1,m} \cup S_{n-m-1}$ is the unique bipartite graph with the maximum sum of squares of degrees in $\mathscr{B}(n, m)$. So we have $B^* \cong K_{1,m} \cup S_{n-m-1}$.

Case 2. $m = n-1$.

In this case we have $B^l(n, m, k_0) = K_{1,n-1}$. Therefore, $\sigma_2(K_{1,n-1}) = \sigma_2(B^*)$. If $B^* \ncong K_{1,n-1}$, then

$$\sigma_2(K_{1,n-1} \cup S_1) = \sigma_2(K_{1,n-1}) = \sigma_2(B^*) = \sigma_2(B^* \cup S_1),$$

which is a contradiction to the result in the Case 1. □

Theorem 2. *Let n, m be two integers with $n \geq 2$, $n \leq m \leq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ and $k_0 = max\{k|m = qk + r, 0 \leq r < k, \lceil \frac{n}{2} \rceil \leq k \leq n - q - sgn(r)\}$. Suppose that $\sigma_2(B^*)$ attains the maximum value among all the graphs in $\mathscr{B}(n, m)$. Then*
(a) $B^* \cong B^l(n, m, k_0)$, or $B^l(n, m, n-k_0)$ if $m > (n-k_0)(k_0-1)$;
(b) $B^* \cong B^l(n, m, k_0)$, $B^l(n, m, n-k_0)$, or $B^l(n, m, k_0-1)$ if $m = (n-k_0)(k_0-1)$;
(c) $B^* \cong B^l(n, m, k_0)$ if $m < (n-k_0)(k_0-1)$.

Proof. Let $m = q_0 k_0 + r_0 = q'_0(k_0+1) + r'_0$, where $0 \leq r_0 < k_0$, $0 \leq r'_0 < k_0+1$. We first prepare three claims.

Claim 1. $m > (k_0+1)(n-k_0-1)$.

Proof. Suppose that $m \le (k_0+1)(n-k_0-1)$. Then $B^l(n,m,k_0+1)$ exists in $\mathscr{B}(n,m,k_0+1)$. This implies that $k_0+1 \le n-q_0'-sgn(r_0')$, contradicting the maximum of k_0. $\square$

Claim 2. There exist no isolated vertices in $B^l(n,m,k_0)$.

Proof. Suppose that there exists an isolated vertex in $B^l(n,m,k_0)$. Since $n \le m$, we have $q_0 \ge 1$. Let (X_0,Y_0) be the bipartition of $B^l(n,m,k_0)$ with $|X_0| = k_0$. Then by the definition of $B^l(n,m,k_0)$, the isolated vertex must be in Y_0. Hence we have $m \le k_0(n-k_0-1) \le (k_0+1)(n-k_0-1)$, contradicting Claim 1. $\square$

Let $k \ge \lceil \frac{n}{2} \rceil$ be an integer. Suppose that $m = qk+r = q'(k+1)+r'$, where $0 \le r < k$, $0 \le r' < k+1$. Then we have $q = \lfloor \frac{m}{k} \rfloor$ and $q' = \lfloor \frac{m}{k+1} \rfloor$.

Claim 3. $\lfloor \frac{m}{k} \rfloor - \lfloor \frac{m}{k+1} \rfloor \le 1$.

Proof. If $\lfloor \frac{m}{k} \rfloor - \lfloor \frac{m}{k+1} \rfloor \ge 2$, then

$$\begin{aligned} r' &= \lfloor \frac{m}{k} \rfloor k + r - \lfloor \frac{m}{k+1} \rfloor (k+1) \\ &\ge \lfloor \frac{m}{k} \rfloor k + r - (\lfloor \frac{m}{k} \rfloor - 2)(k+1) \\ &= r + 2(k+1) - \lfloor \frac{m}{k} \rfloor \\ &\ge r + 2(k+1) - k \\ &> k+1, \end{aligned}$$

a contradiction. $\square$

By the definition of $B^l(n,m,k)$, we have

$$\begin{aligned} \sigma_2(B^l(n,m,k)) &= r(q+1)^2 + (k-r)q^2 + qk^2 + r^2 \\ &= (m-qk)(q+1)^2 + (k+qk-m)q^2 + qk^2 + (m-qk)^2 \\ &= q(k-1)(k+qk-2m) + m^2 + m \\ &= \lfloor \frac{m}{k} \rfloor (k-1)(k + \lfloor \frac{m}{k} \rfloor k - 2m) + m^2 + m. \end{aligned}$$

Set $f(k) = \sigma_2(B^l(n,m,k))$. Then

$$f(k+1) - f(k) = \lfloor \frac{m}{k+1} \rfloor k(k+1+\lfloor \frac{m}{k+1} \rfloor (k+1) - 2m) - \lfloor \frac{m}{k} \rfloor (k-1)(k + \lfloor \frac{m}{k} \rfloor k - 2m).$$

If $\lfloor \frac{m}{k} \rfloor - \lfloor \frac{m}{k+1} \rfloor = 0$, then

$$f(k+1) - f(k) = 2\lfloor \frac{m}{k} \rfloor (\lfloor \frac{m}{k} \rfloor k + k - m) > 0. \tag{1}$$

If $\lfloor\frac{m}{k}\rfloor - \lfloor\frac{m}{k+1}\rfloor = 1$, then

$$f(k+1) - f(k) = 2(\lfloor\frac{m}{k}\rfloor - k)(\lfloor\frac{m}{k}\rfloor k - m) \geq 0. \quad (2)$$

Thus, $f(k)$ is a nondecreasing function. So we have

$$f(k_0) \geq f(k_0 - 1) \geq f(k_0 - 2) \geq \cdots \geq f(\lceil\frac{n}{2}\rceil). \quad (3)$$

By Lemma 1, we know that $\sigma_2(B^*) = \max\{f(k_0), f(k_0-1), \ldots, f(\lceil\frac{n}{2}\rceil)\}$. Let (X^*, Y^*) be the bipartition of B^* with $|X^*| \geq \lceil n/2\rceil$. We distinguish two cases.

Case 1. $k_0 = \lceil\frac{n}{2}\rceil$.

First, we have $n = 2k_0$ or $2k_0 - 1$. It is clear that

$$m \leq k_0(n - k_0). \quad (4)$$

Suppose that $n = 2k_0$. Then by Claim 1 and (4) we have

$$k_0^2 - 1 < m \leq k_0^2,$$

i.e., $m = k_0^2$. This means that $B^l(n, m, k_0)$ is the unique graph in $\mathscr{B}(n, m)$. So we have $B^* \cong B^l(n, m, k_0)$.

Suppose that $n = 2k_0 - 1$. Then by Claim 1 and (4) we have

$$(k_0 + 1)(k_0 - 2) < m \leq k_0(k_0 - 1).$$

This implies that $m = k_0(k_0 - 1)$ or $k_0(k_0 - 1) - 1$. In either cases, $B^l(n, m, k_0)$ is the unique graph in $\mathscr{B}(n, m)$. So we have $B^* \cong B^l(n, m, k_0)$.

Case 2. $k_0 > \lceil\frac{n}{2}\rceil$.

Case 2.1. $f(k_0) = f(k_0 - 1)$.

Let $m = q_0''(k_0 - 1) + r_0'' = q_0'''(k_0 - 2) + r_0'''$, where $0 \leq r_0'' < k_0 - 1$, $0 \leq r_0''' < k_0 - 2$. Then we have $q_0'' = \lfloor\frac{m}{k_0-1}\rfloor$ and $q_0''' = \lfloor\frac{m}{k_0-2}\rfloor$.

Since $f(k_0) = f(k_0 - 1)$, it follows from (1) and (2) that

$$f(k_0) - f(k_0 - 1) = 2(\lfloor\frac{m}{k_0 - 1}\rfloor - (k_0 - 1))(\lfloor\frac{m}{k_0 - 1}\rfloor(k_0 - 1) - m) = 0.$$

So we have

$$\lfloor\frac{m}{k_0 - 1}\rfloor - (k_0 - 1) = 0 \quad \text{or} \quad \lfloor\frac{m}{k_0 - 1}\rfloor(k_0 - 1) - m = 0.$$

Suppose that $\lfloor\frac{m}{k_0-1}\rfloor - (k_0 - 1) = 0$. Since $k_0 - 1 \geq \lceil\frac{n}{2}\rceil$, we have

$$m \geq (k_0 - 1)^2 \geq (\lceil\frac{n}{2}\rceil)^2.$$

By the condition $m \leq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$, we can easily deduce that $m = (k_0 - 1)^2$. Again, with $k_0 - 1 \geq \lceil \frac{n}{2} \rceil$, we have

$$m = (k_0 - 1)^2 > k_0(k_0 - 2) \geq k_0(n - k_0),$$

a contradiction.

Suppose that $\lfloor \frac{m}{k_0-1} \rfloor (k_0 - 1) - m = 0$. Then we have $r_0'' = 0$. Since $f(k_0) = f(k_0 - 1)$, by (1) and (2) we can conclude that $\lfloor \frac{m}{k_0-1} \rfloor - \lfloor \frac{m}{k_0} \rfloor = 1$.

Suppose that $r_0 = 0$. Then

$$m = q_0 k_0 = \lfloor \frac{m}{k_0 - 1} \rfloor (k_0 - 1) = (q_0 + 1)(k_0 - 1).$$

This implies that $q_0 = k_0 - 1$. It follows from Claim 2 that $k_0 = \lceil \frac{n}{2} \rceil$, a contradiction.

Suppose $r_0 \neq 0$. Then by Claim 2, we can conclude that $k_0 + q_0 + 1 = n$. So we have

$$m = \lfloor \frac{m}{k_0 - 1} \rfloor (k_0 - 1) = (\lfloor \frac{m}{k_0} \rfloor + 1)(k_0 - 1) = (n - k_0)(k_0 - 1). \tag{5}$$

Suppose that $k_0 - 2 \geq \lceil \frac{n}{2} \rceil$ and $f(k_0) = f(k_0 - 1) = f(k_0 - 2)$. Then it follows from (1) and (2) that

$$f(k_0 - 1) - f(k_0 - 2) = 2(\lfloor \frac{m}{k_0 - 2} \rfloor - (k_0 - 2))(\lfloor \frac{m}{k_0 - 2} \rfloor (k_0 - 2) - m) = 0.$$

As the proof of $\lfloor \frac{m}{k_0-1} \rfloor - (k_0 - 1) \neq 0$ for the case $f(k_0) = f(k_0 - 1)$, we can prove that $\lfloor \frac{m}{k_0-2} \rfloor - (k_0 - 2) \neq 0$. So let us now assume that $\lfloor \frac{m}{k_0-2} \rfloor (k_0 - 2) - m = 0$. Then we have $r_0''' = 0$. Since $f(k_0 - 1) = f(k_0 - 2)$, by (1) and (2) we can conclude that $\lfloor \frac{m}{k_0-2} \rfloor - \lfloor \frac{m}{k_0-1} \rfloor = 1$. Then, by (5), we have

$$m = (n - k_0)(k_0 - 1) = \lfloor \frac{m}{k_0 - 2} \rfloor (k_0 - 2) = (n - k_0 + 1)(k_0 - 2).$$

This implies that $n = 2k_0 - 2$, contradicting our assumption $k_0 - 2 \geq \lceil \frac{n}{2} \rceil$.

Therefore, we have $f(k_0) = f(k_0 - 1) > f(k_0 - 2)$. This means that $B^* \in \mathscr{B}(n, m, k_0)$ or $\mathscr{B}(n, m, k_0 - 1)$.

Suppose that $B^* \in \mathscr{B}(n, m, k_0)$. Then $\sigma_2(B^*)$ attains the maximum value among all the graphs in $\mathscr{B}(n, m, k_0)$. Note that $m = (n - k_0)(k_0 - 1)$. So we have $k_0(n - k_0) - m = n - k_0$. It follows from Lemma 4 that $\sigma_2(\overline{B^*})$ attains the maximum value among all the graphs in $\mathscr{B}(n, n-k_0, k_0)$. By Theorem 1, we obtain that $\overline{B^*} \cong K_{1,n-k_0} \cup S_{k_0-1}$. If the $n-k_0$ pendent vertices of $\overline{B^*}$ are in X^*, then by Lemma 4, we have $B^* \cong B^l(n, m, k_0)$. If the $n - k_0$ pendent vertices of $\overline{B^*}$ are in Y^*, then by Lemma 4, we have $B^* \cong B^l(n, m, n - k_0)$, which ia also a graph in $\mathscr{B}(n, m, k_0)$

Suppose that $B^* \in \mathscr{B}(n, m, k_0 - 1)$. Then $\sigma_2(B^*)$ attains the maximum value among all the graphs in $\mathscr{B}(n, m, k_0 - 1)$. Note that $m = (n - k_0)(k_0 - 1)$. Then we have $(k_0 - 1)(n - k_0 + 1) - m = k_0 - 1$. It follows from Lemma 4 that $\sigma_2(\overline{B^*})$ attains the maximum value among all the graphs in $\mathscr{B}(n, k_0 - 1, k_0 - 1)$. By Theorem 1, we obtain that $\overline{B^*} \cong K_{1,k_0-1} \cup S_{n-k_0}$. Since $k_0 - 1 \geq \lceil \frac{n}{2} \rceil$, we have $k_0 - 1 \geq n - k_0 + 1$. So all the pendent vertices are in X^*. By Lemma 4, we have $B^* \cong B^l(n, m, k_0 - 1)$.

Case 2.2. $f(k_0) > f(k_0 - 1)$.

In this case, we have $B^* \in \mathscr{B}(n, m, k_0)$ and $\sigma_2(B^*)$ attains the maximum value among all the graphs in $\mathscr{B}(n, m, k_0)$. From Claim 3 we know that $\lfloor \frac{m}{k_0-1} \rfloor - \lfloor \frac{m}{k_0} \rfloor \leq 1$. Suppose $\lfloor \frac{m}{k_0-1} \rfloor - \lfloor \frac{m}{k_0} \rfloor = 0$. Then we have $q_0'' = q_0$ and $r_0'' = q_0 + r_0$. By $r_0'' < k_0 - 1$, we get $r_0 < k_0 - q_0 - 1$. If $r_0 = 0$, then

$$m = (n - k_0)k_0 > (n - k_0)(k_0 - 1).$$

If $r_0 > 0$, then

$$\begin{aligned} m &= (n - k_0 - 1)k_0 + r_0 \\ &= (n - k_0)(k_0 - 1) + n - 2k_0 + r_0 \\ &< (n - k_0)(k_0 - 1). \end{aligned}$$

Suppose $\lfloor \frac{m}{k_0-1} \rfloor - \lfloor \frac{m}{k_0} \rfloor = 1$. Then $m - \lfloor \frac{m}{k_0-1} \rfloor (k_0 - 1) > 0$. Since $\lfloor \frac{m}{k_0-1} \rfloor \geq n - k_0$, we have

$$m > (n - k_0)(k_0 - 1).$$

Therefore, in the following we consider two subcases.

Case 2.2.1. $m > (n - k_0)(k_0 - 1)$.

By Lemma 4 we know that $\sigma_2(\overline{B^*})$ attains the maximum value among all the graphs in $\mathscr{B}(n, k_0(n - k_0) - m, k_0)$. Since $k_0(n - k_0) - m < n - k_0 \leq n - 1$, it follows from Theorem 1 that $B^* \cong k_{1,k_0(n-k_0)-m} \cup S_{n-k_0(n-k_0)+m-1}$. If the $k_0(n - k_0) - m$ pendent vertices are in X^*, then by Lemma 4, we have $B^* \cong B^l(n, m, k_0)$. If the $k_0(n - k_0) - m$ pendent vertices are in Y^*, then by Lemma 4, we have $B^* \cong B^l(n, m, n - k_0)$, which ia also a graph in $\mathscr{B}(n, m, k_0)$.

Case 2.2.2. $m < (n - k_0)(k_0 - 1)$.

It follows from Lemma 4 that $\sigma_2(\overline{B^*})$ attains the maximum value among all the graphs in $\mathscr{B}(n, k_0(n - k_0) - m, k_0)$. By Claim 1, we can conclude that $k_0(n - k_0) - m < 2k_0 -$

$n+1 \le n-1$. Then by Theorem 1 we have $B^* \cong k_{1,k_0(n-k_0)-m} \cup S_{n-k_0(n-k_0)+m-1}$. Since $k_0(n-k_0)-m > n-k_0$, we know that the $k_0(n-k_0)-m$ pendent vertices are in X^*. By Lemma 4, we have $B^* \cong B^l(n,m,k_0)$.

The proof is complete. $\square$

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